Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-02-05

Problem 1. Consider the line through the points $A(1, 3, 2)$, $B(3, -1, 6)$, and the line through the points $C(5, 2, 0)$, $D(3, 6, -4)$. Are these lines skew?

Problem 2. Let Q, R be points on a line L. Explain why the distance from some point P to L is

$$\frac{|\overrightarrow{QR} \times \overrightarrow{QP}|}{|\overrightarrow{QR}|},$$

perhaps by interpreting the numerator and denominator of this quantity geometrically.

Given four points P, Q, R, S, can you come up with a similar formula for the distance from the point P to the plane containing Q, R, S (assuming they are not collinear)?

Problem 3. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

Problem 4. Are there vectors $\mathbf{v}$ such that $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$? If so, find all of them. If not, explain why not.

Then do the same question, but for $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, 5\rangle$.

Below are the numerical answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers

Problem 1. No, they are not. They are parallel.

Problem 2. Consider the parallelogram with adjacent sides QR and QP. The numerator is the area of this parallelogram, and the denominator can be interpreted as the base, in which case the ratio equals the "height" measured perpendicularly to that base. That is none other than the desired distance.

For the distance to the plane:

$$\frac{|\overrightarrow{QP} \cdot (\overrightarrow{QR} \times \overrightarrow{QS})|}{|\overrightarrow{QR} \times \overrightarrow{QS}|}$$

would work.

Problem 3. You could write out all the components and check that way, but it gets really messy. It is faster to do the following:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2. \end{aligned}$$

Problem 4. The most straightforward way to find all such vectors $\mathbf{v}$ is just to solve the system of equations: writing $\mathbf{v} = \langle a, b, c \rangle$ we have

$$\begin{aligned} 2c - b &= 3 \\ a - c &= 1 \\ b - 2a &= -5. \end{aligned}$$

Some algebra will show you that there are actually only two equations (one is redundant), thus you will have one free variable. All the solutions can be represented by

$$\mathbf{v} = \langle a, 2a - 5, a - 1 \rangle$$

for some real number a. (This is not the only way to write it.)

On the other hand, there are no vectors $\mathbf{v}$ for which $\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$. This is because $\langle 1, 2, 3 \rangle \cdot \langle 3, 1, 5 \rangle \neq 0$. (You would also see this if you tried to solve the system of equations, which will be inconsistent.)